THE UNIVERSITY OF BRITISH COLUMBIA

# Approximate Normalization for Gradual Dependent Types

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*University of British Columbia, University of Chile/INRIA*
*ICFP 2019*

# Our Contributions

# GDTL:
# Gradual Dependently Typed Language

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- Full spectrum, universe hierarchy

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- Decidable typechecking

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This talk

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- Decidable typechecking

This talk

- Proof of gradual type safety

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- Proof of gradual type safety
- *Gradual Guarantees* (Siek et al 2015): reducing precision of term won't create new static or dynamic failures

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# Gradual Dependently Typed Language

- Full spectrum, universe hierarchy
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This talk

- Proof of gradual type safety
- *Gradual Guarantees* (Siek et al 2015): reducing precision of term won't create new static or dynamic failures

See paper

# Why Gradual Dependent Types?

# Dependent Types: Pain and Promise

# Expectation

```
$> compile ./myprogram
>> 0 bugs detected!
```

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## Reality

```
$> compile ./myprogram
>> Type mismatch between
        Vec Nat (m+n)
    and
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- Popular for proof assistants

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## Reality

```
$> compile ./myprogram
>> Type mismatch between
       Vec Nat (m+n)
   and
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```

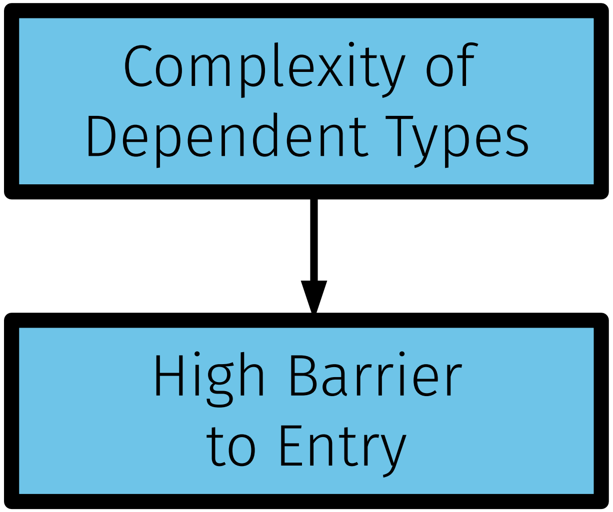
- Popular for proof assistants
- Not popular for programming

# A Vicious Cycle

# A Vicious Cycle

Complexity of Dependent Types

# A Vicious Cycle



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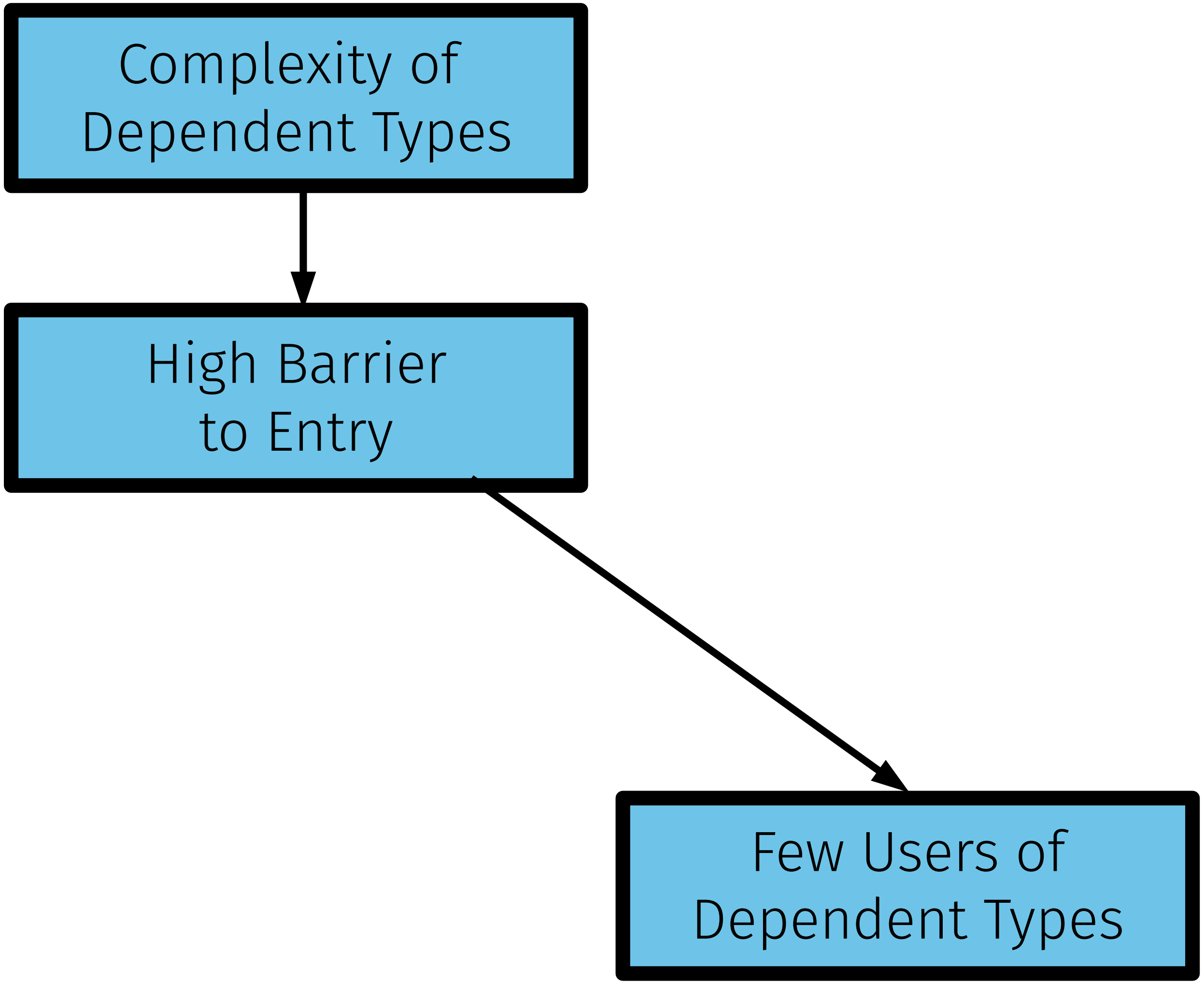

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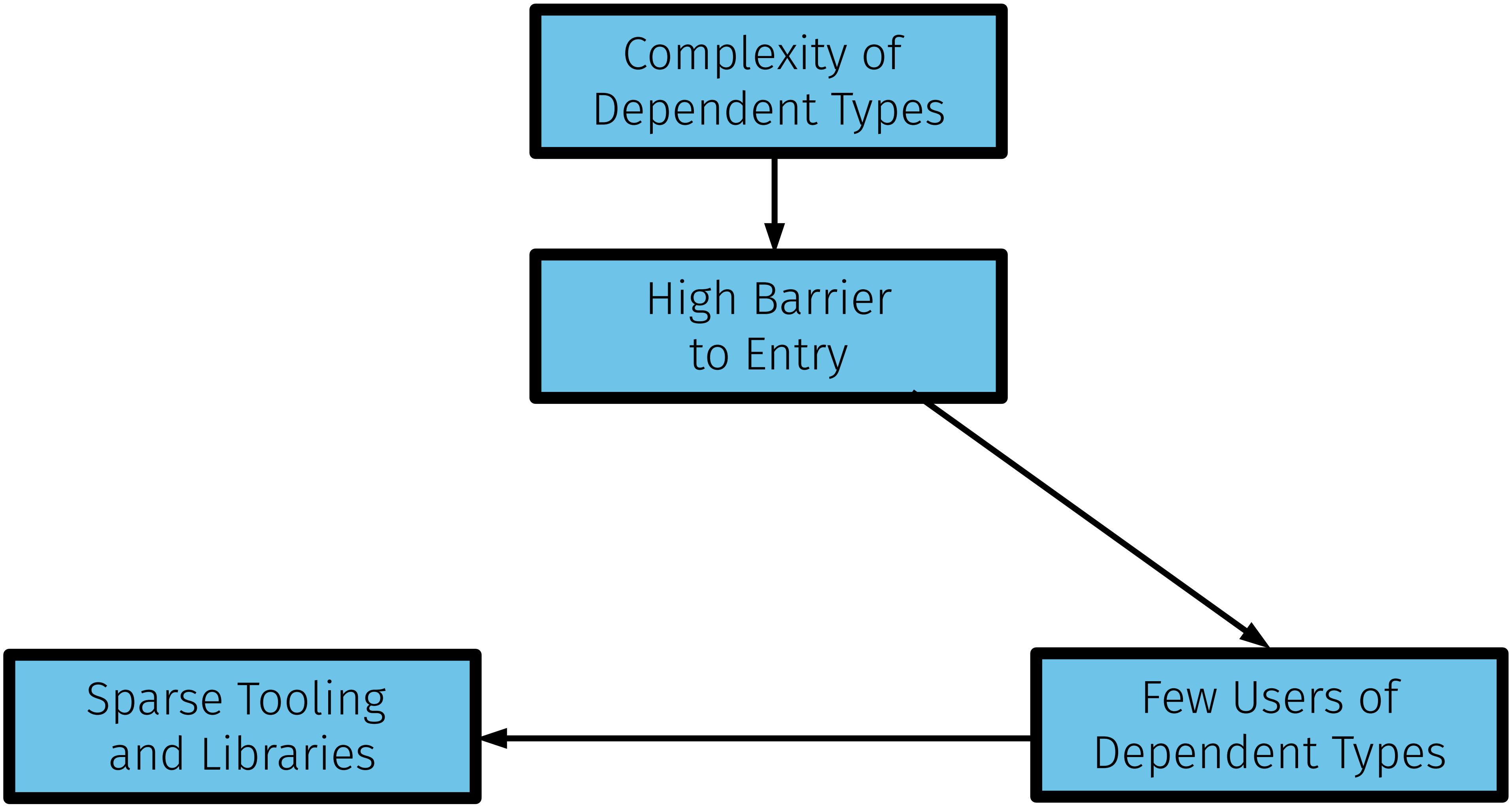

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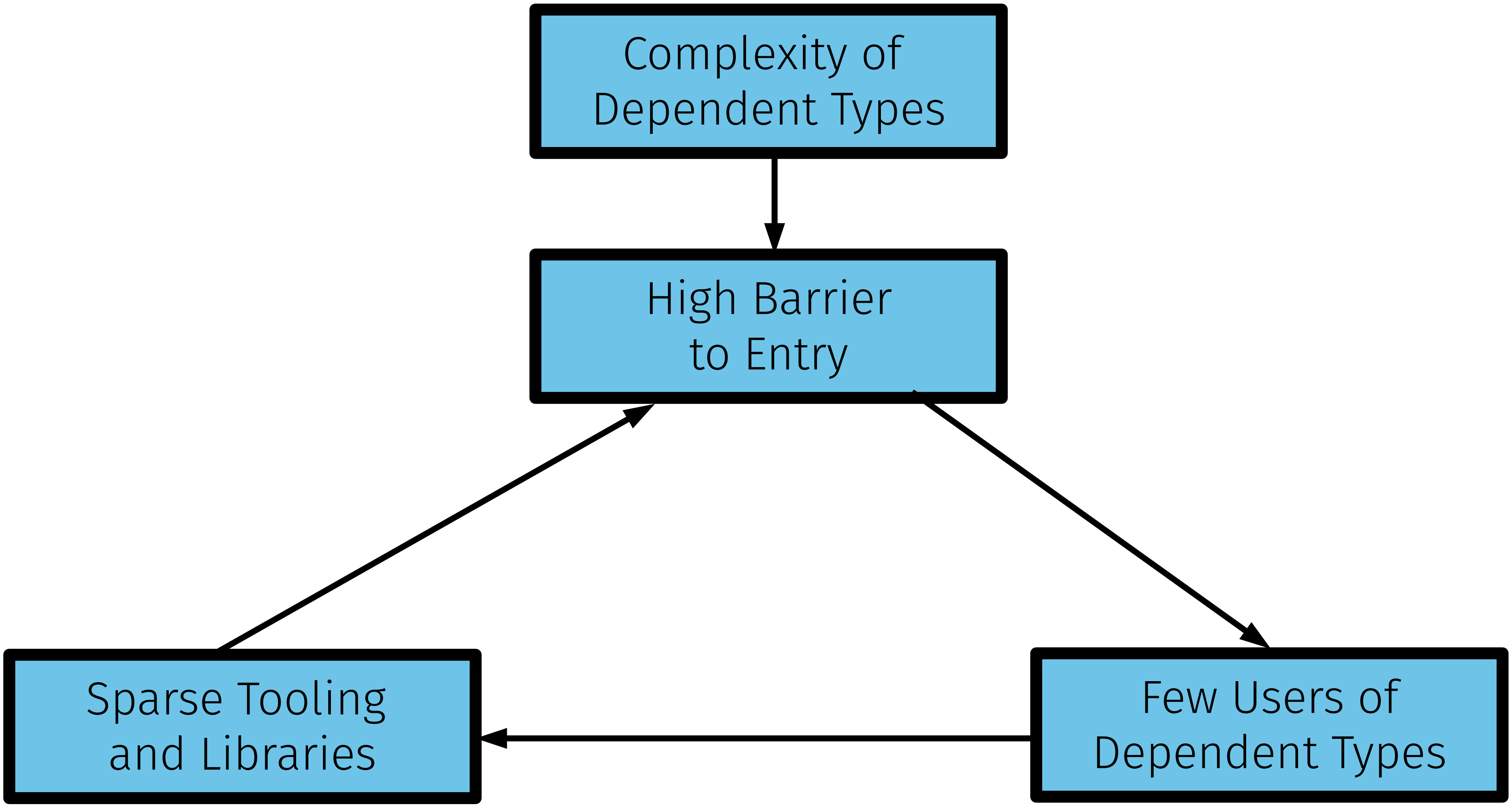

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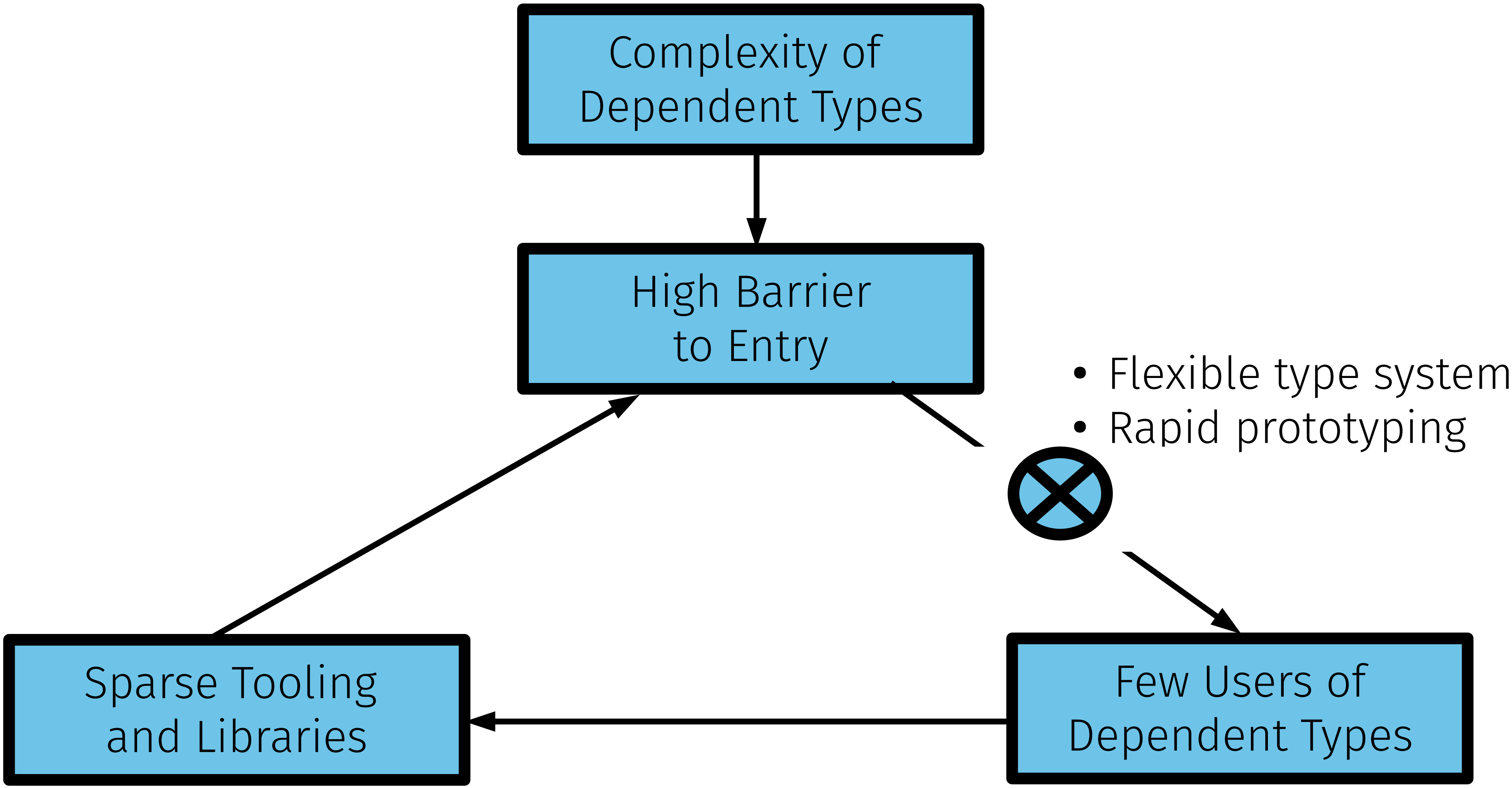

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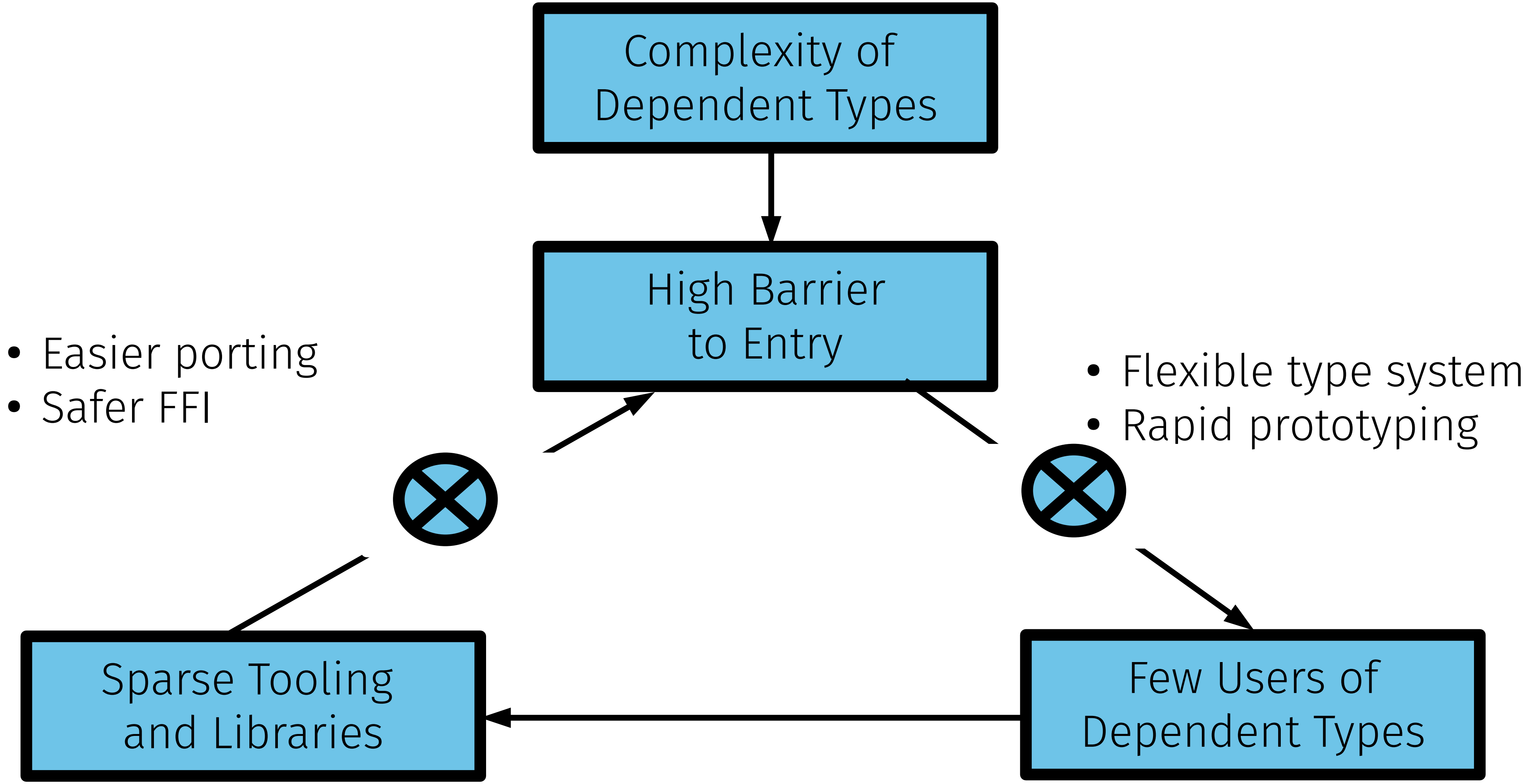

# Goals For
# Gradual Dependent Types

# Motivation: Lists vs. Vectors

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```
data List a
  where
    Nil : List a
    Cons : a
         -> List a
         -> List a

head : List a -> a
```

No size knowledge
in type

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Error on `Nil`

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```
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```

```
data Vec (a : Type) (n : Nat)
 where
    Nil : Vec a 0
    Cons : a
         -> Vec a n
         -> Vec a (n + 1)

head : Vec a (n + 1) -> a
```

Error on `Nil`

# Motivation: Lists vs. Vectors

No size knowledge in type

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data List a
  where
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Error on `Nil`

Length in *type index*

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 where
    Nil  : Vec a 0
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```

```
head : Vec a (n + 1) -> a
```

Won't typecheck for `Nil`

# Porting Code to Dependent Types

```
sort : List Int -> List Int
```

## Porting Code to Dependent Types

```
sort : List Int -> List Int

sort Nil = Nil
```

```
sort : List Int -> List Int
sort Nil = Nil
sort (Cons head tail) =
```

```
sort : List Int -> List Int

sort Nil = Nil

sort (Cons head tail) =
  sort (filter (<= head) tail))
```

```
sort : List Int -> List Int

sort Nil = Nil

sort (Cons head tail) =
  sort (filter (<= head) tail))
  ++ [head]
```

```
sort : List Int -> List Int

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filter : (Int -> Bool)
       -> List Int -> List Int
```

```
       Vec Int n
sort : List Int -> List Int

sort Nil = Nil

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```

```
       Vec Int n    Vec Int n
sort : List Int -> List Int

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         Vec Int n    Vec Int n
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```

Need proof that lengths sum to n

```
filter : (Int -> Bool)
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```

# Porting Code to Dependent Types

`Vec Int n` `Vec Int n`

`sort :` ~~`List Int`~~ `->` ~~`List Int`~~

```
sort Nil = Nil ✓

sort (Cons head tail) =
  sort (filter (<= head) tail))
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Need proof that lengths sum to n

```
filter : (Int -> Bool)
```

`->` ~~`List Int`~~ `-> List Int`

`Vec Int n`

# Porting Code to Dependent Types

```
          Vec Int n    Vec Int n
sort : List Int -> List Int

sort Nil = Nil ✓

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  sort (filter (<= head) tail))
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```

Need proof that lengths sum to n

```
filter : (Int -> Bool)
         -> List Int -> List Int
            Vec Int n   Vec Int ___
```

# How To Solve?

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| | | |
|---|---|---|
| Static Dependent Types | | |
| | | |
| | | |

# How To Solve?

| | | |
|---|---|---|
| Static Dependent Types | Existential Types, Inductive Proof | |
| | | |
| | | |

# The Static Approach

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```
rewriteFilterLength :
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```
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  (v : Vec Int n)
  -> (p : Int -> Bool)
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## The Static Approach

```
rewriteFilterLength :
  (v : Vec Int n)
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  -> Vec Int
       (length (filter p v)
        + 1 + length (filter (not . p) v))
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rewriteFilterLength :
  (v : Vec Int n)
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        (length (filter p v)
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Relies on induction, commutativity, etc.

```
sort : Vec Int n -> Vec Int n

sort Nil = Nil

sort (Cons head tail) =

  sort (filter (<= head) tail))
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  ++ sort (filter (> head) tail))


filter : (Int -> Bool)
       -> Vec Int n ->   Vec Int ?
```

# Gradual Proof Terms

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sort (Cons head tail) =
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| Static Dependent Types | Existential Types, Inductive Proof | |
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## How To Solve?

| Static Dependent Types | Existential Types, Inductive Proof | ✗ Significant effort required |
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| Static Dependent Types | Existential Types, Inductive Proof | ✗ Significant effort required |
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| Gradual Dependent Types | | |

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| Gradual Dependent Types | `filter` returns `Vec Int ?` unknown *length* | ✓ Precise in type, flexible in length! |

Our approach!

# The GDTL Solution

```
sort : Vec Int n -> Vec Int n

sort Nil = Nil

sort (Cons head tail) =
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  ++ [head]
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filter : (Int -> Bool)
       -> Vec Int n ->
```

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filter : (Int -> Bool)
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# The GDTL Solution

```
sort : Vec Int n -> Vec Int n

sort Nil = Nil

sort (Cons head tail) =
  sort (filter (<= head) tail))
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```

**?**+1+**?** evals to **?**, is *consistent* with n

```
filter : (Int -> Bool)
       -> Vec Int n -> Vec Int ?
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# Gradual Type Safety

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```
head : Vec a (n+1) -> a
```

```
head : Vec a (n+1) -> a

x : Vec a 0
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head : Vec a (n+1) -> a

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theHead = head x
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head : Vec a (n+1) -> a

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- Does not typecheck

# Gradual Type Safety

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head : Vec a (n+1) -> a

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head : Vec a (n+1) -> a

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- Typechecks!

```
head : Vec a (n+1) -> a

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```
x ↦ Nil
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```
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x ↦ Nil

- Runtime error

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x ↦ Cons 1 Nil

# Gradual Type Safety

```
head : Vec a (n+1) -> a

x : Vec a ?

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```

x ↦ 

- Runs successfully

x ↦ Cons 1 Nil

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Need proof that lengths sum to n

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filter : (Int -> Bool)
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# Filling in the Proof

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rewriteFilterLength :
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```
rewriteFilterLength = ?
```

Like Idris/Agda typed holes

```
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filter : (Int -> Bool)
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```

This code typechecks *and runs!*

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head ((subst badProof nil) :: Vec Int 1)
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- **?** has type **?**, can use at any type
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- Typechecks
- Runtime type error

# GDTL: A Gradual Dependently Typed Language

Statics + Dynamics mostly using
*Abstracting Gradual Typing (Garcia et. al. 2016)*

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**Main extensions:**

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Type/Term Overlap

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Type/Term Overlap $\longrightarrow$ **?** as unknown type *and* term

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Type Indices

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Proof term

Statics + Dynamics mostly using
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**Main extensions:**

Type/Term Overlap ⟶ **?** as unknown type *and* term

Type Indices ⟶ **?** as type index

Proof term ⟶ **?** as a term *at runtime*

# What’s the Catch?

# Dependent Types

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Evaluate terms
*at compile time*

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Strongly
normalizing

# Dependent Types

Evaluate terms
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Failure free

| Dependent Types | Gradual Types |
|---|---|
| Evaluate terms *at compile time* | |
| Strongly normalizing | |
| Failure free | |

| **Dependent Types** | **Gradual Types** |
|---|---|
| Evaluate terms *at compile time* | Evaluating has effects |
| Strongly normalizing | |
| Failure free | |

## Dependent Types

Evaluate terms
*at compile time*

Strongly
normalizing

Failure free

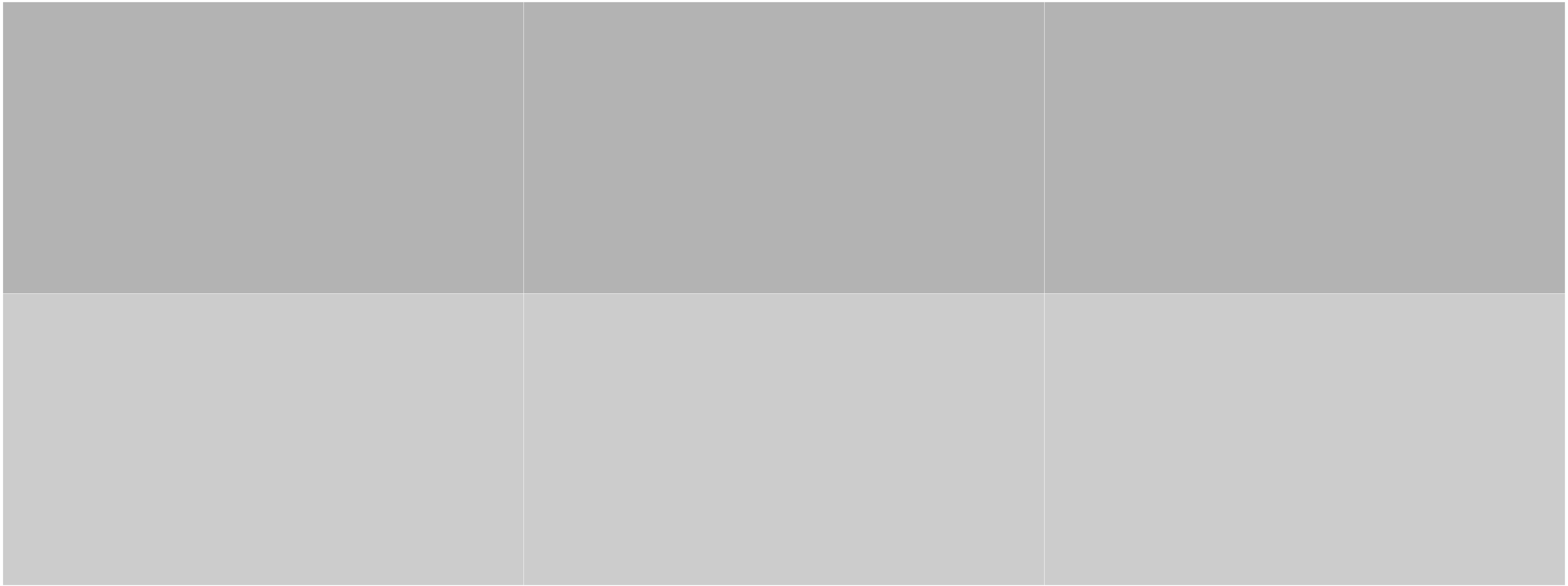
## Gradual Types

Evaluating has
effects

Can diverge
i.e. $\lambda(x : \mathbf{?}).\, x\ x$

| **Dependent Types** | **Gradual Types** |
|---|---|
| Evaluate terms *at compile time* | Evaluating has effects |
| Strongly normalizing | Can diverge i.e. $\lambda(x : \mathbf{?}).\, x\ x$ |
| Failure free | Type errors in evaluation |

# **Key Idea:** Approximate Normalization



## **Key Idea:** Approximate Normalization

Exploit the phase distinction:

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Exploit the phase distinction:

| **Compile-time Normalization** | | |
|---|---|---|
| | | |

Exploit the phase distinction:

| | | |
|---|---|---|
| **Compile-time Normalization** | Always terminates | |
| | | |

# **Key Idea:** Approximate Normalization

Exploit the phase distinction:

| | | |
|---|---|---|
| **Compile-time Normalization** | Always terminates | Approximate results |
| | | |

# **Key Idea:** Approximate Normalization

Exploit the phase distinction:

| | | |
|---|---|---|
| **Compile-time Normalization** | Always terminates | Approximate results |
| **Runtime Evaluation** | | |

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## **Key Idea:** Approximate Normalization

Exploit the phase distinction:

| | | |
|---|---|---|
| **Compile-time Normalization** | Always terminates | Approximate results |
| **Runtime Evaluation** | May diverge | Exact results |

# Compile-Time - Approximation #1: Termination

**Based on *Hereditary Substitution***
(Watkins et al 2003)

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Types structurally decreasing?

## **Compile-Time -** Approximation #1: Termination

**Based on *Hereditary Substitution***
(Watkins et al 2003)

- Static version: normalization is *structurally recursive* on types
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Types structurally decreasing?

***Yes***

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***Yes***

Proceed normally

## Based on *Hereditary Substitution*

(Watkins et al 2003)

- Static version: normalization is *structurally recursive* on types
- Our version:

Types structurally decreasing?

***Yes***

***No***

Proceed normally

## Based on *Hereditary Substitution*

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- Static version: normalization is *structurally recursive* on types
- Our version:

Types structurally decreasing?

***Yes*** → Proceed normally

***No*** → Return **?**

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$$\curlyvee$$

$$z : \mathbf{?} \to \mathbf{?}$$

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$$y : \mathbf{?}$$

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$$
\begin{array}{ll}
(\lambda x.\, x(\lambda y.\, yy))(\lambda z.\, zz) & x : (\mathbf{?} \to \mathbf{?}) \to \mathbf{?} \\
\rotatebox[origin=c]{-90}{$\rightsquigarrow$} & \curlyvee \\
(\lambda z.\, zz)(\lambda y.\, yy) & z : \mathbf{?} \to \mathbf{?} \\
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(\lambda y.\, yy)(\lambda y.\, yy) & y : \mathbf{?}
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# Compile-Time - Approximation #2: Downcasts

Order types by *precision*

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Losing type information : ✓ **Safe**

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Unsafe operation: approximate!

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## **Run-Time -** Execution

- Terms annotated with *evidence*
  - Most-precise currently-known type info
- Combined using precision-meet
  - Runtime error if meet does not exist

# Wrapping Up

# What We Built

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**GDTL:**

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- Blame and Error Reporting
- Eventual Goal: Idris frontend

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